The ∞-Categorical Yoneda Lemma

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Abstract

We give a structural proof of the ∞-categorical Yoneda lemma in the quasicategorical model. Representable presheaves are treated through their associated slice left fibrations, the space of sections of a pullback left fibration is identified with the limit of its straightening, and the resulting limit is computed by restriction along the final inclusion of the terminal object of the slice ∞-category. In this form, the Yoneda equivalence appears as a formal consequence of straightening–unstraightening, pullback stability of left fibrations, and finality.

Note to the Reader

The theorem considered here is not approached as an identity internal to a presheaf category. Its natural setting is the geometry of left fibrations. The representable presheaf attached to an object C is encoded by the slice left fibration $\mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{C}^{\mathrm{op}}$, and the passage from morphisms into an arbitrary presheaf to evaluation at C is governed by the universal properties of pullback, section, limit, and final restriction. The point of view adopted below is therefore structural: representability is expressed geometrically, and Yoneda is recovered from the formal infrastructure of ∞-category theory.

1 Introduction

In the ∞-categorical setting, the Yoneda lemma is most naturally read as a statement about representable left fibrations. The presheaf $j(C)$ represented by an object $C \in \mathcal{C}$ is classified by the slice projection $\mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{C}^{\mathrm{op}}$, and a map from $j(C)$ to a presheaf F is thereby identified, via straightening–unstraightening, with a section of the pullback of the left fibration classified by F along this slice projection. The resulting space of sections is a limit of spaces, and that limit is computed by restriction to the terminal object id_C of the slice ∞-category.

The organization of the paper follows this formal pattern. After fixing conventions, representable left fibrations are recalled, the universal identification of maps with sections and sections with limits is isolated, and finality in the slice is established. The Yoneda equivalence then follows by composition of these structures. No change in categorical level occurs anywhere in the argument: the proof is entirely internal to the quasicategorical model and is expressed in homotopy-coherent terms throughout.

2 Conventions

We work in the quasicategorical model of ∞-categories. Thus an ∞-category is a simplicial set satisfying the inner horn-filling condition. Equivalences are equivalences in the sense of ∞-categories; limits are limits in the ambient ∞-categorical sense, hence homotopy limits; mapping spaces are the canonical Kan complexes attached to an ∞-category. Fix Grothendieck universes $\mathcal{U} \in \mathcal{V}$. The adjective "small" means "$\mathcal{U}$-small." The ∞-category $\mathcal{S}$ of spaces is taken in $\mathcal{V}$, with the full subcategory of $\mathcal{U}$-small spaces understood implicitly. Consequently, if $\mathcal{C}$ is a small ∞-category, then

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

is a $\mathcal{V}$-small ∞-category.

For an ∞-category K, we write $\mathrm{LFib}(K)$ for the full subcategory of $\mathrm{Cat}_{\infty,/K}$ spanned by left fibrations over K. If $p\colon X \to K$ is a left fibration and $k \in K$ is an object, its fiber at k means the pullback

$$X_k := X \times_K \{k\},$$

regarded as a space. If $r\colon X \to K$ is any morphism in Cat_∞, we write

$$\Gamma(r) := \mathrm{Map}_{\mathrm{Cat}_{\infty,/K}}(\mathrm{id}_K, r)$$

for the space of sections of r.

We fix once and for all the straightening–unstraightening equivalence in the form

$$\mathrm{St}_K\colon \mathrm{LFib}(K) \xrightarrow{\simeq} \mathrm{Fun}(K, \mathcal{S}), \qquad \mathrm{Un}_K\colon \mathrm{Fun}(K, \mathcal{S}) \xrightarrow{\simeq} \mathrm{LFib}(K),$$

functorial in K; cf. HTT, §2.2.1. All identifications below are taken with respect to this equivalence and the universal properties it exhibits.

3 Representable left fibrations

Let $\mathcal{C}$ be a small ∞-category. For $C \in \mathcal{C}$, define the representable presheaf

$$j(C) := \mathrm{Map}_{\mathcal{C}}(-, C) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

This construction is functorial in C and determines the Yoneda embedding

$$j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}).$$

The geometric content of representability is the following standard fact.

Proposition 3.1. *For each object $C \in \mathcal{C}$, the projection*

$$q_C\colon \mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{C}^{\mathrm{op}}$$

is a left fibration, and there is a canonical equivalence

$$\mathrm{St}_{\mathcal{C}^{\mathrm{op}}}(q_C) \simeq j(C).$$

Moreover, the object $\mathrm{id}_C \in \mathcal{C}^{\mathrm{op}}_{/C}$ is terminal: for every object $x \in \mathcal{C}^{\mathrm{op}}_{/C}$, the mapping space

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}_{/C}}(x, \mathrm{id}_C)$$

is contractible.

Proof. The first assertion is the representability statement attached to straightening–unstraightening for slice left fibrations; equivalently, for each object $D \in \mathcal{C}$, the fiber of q_C at the corresponding object of $\mathcal{C}^{\mathrm{op}}$ is canonically equivalent to $\mathrm{Map}_{\mathcal{C}}(D, C)$, compatibly with transport along edges of $\mathcal{C}^{\mathrm{op}}$. This identifies $\mathrm{St}_{\mathcal{C}^{\mathrm{op}}}(q_C)$ with $j(C)$.

For terminality, let $x \in \mathcal{C}^{\mathrm{op}}_{/C}$. By the definition of the slice ∞-category, the mapping space from x to id_C is the fiber of the map from the space of 2-simplices of $\mathcal{C}$ with terminal vertex C to the space of pairs of composable edges having composite the edge corresponding to x, taken at the degenerate factorization of that edge through id_C. This fiber is contractible. Hence $\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}_{/C}}(x, \mathrm{id}_C)$ is contractible. □

4 Sections and limits

The Yoneda lemma will be obtained from two formal identifications.

Proposition 4.1. *Let $q\colon X \to K$ and $p\colon Y \to K$ be morphisms in Cat_∞, and let*

$$\begin{array}{ccc} Y \times_K X & \longrightarrow & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[q]{} & K \end{array}$$

be a pullback square. Then the universal property of the pullback induces a canonical equivalence of spaces

$$\mathrm{Map}_{\mathrm{Cat}_{\infty,/K}}(q, p) \xrightarrow{\simeq} \Gamma(p_X).$$

If p is a left fibration, then so is p_X.

Proof. A point of $\mathrm{Map}_{\mathrm{Cat}_{\infty,/K}}(q, p)$ is, by definition, a morphism $\alpha\colon X \to Y$ in Cat_∞ together with an equivalence $p \circ \alpha \simeq q$ in $\mathrm{Fun}(X, K)$. By the universal property of the pullback, such data are equivalent to a morphism $\widetilde{\alpha}\colon X \to Y \times_K X$ over X, that is, to a section of p_X. This yields the displayed equivalence. Stability of left fibrations under pullback is standard. □

Proposition 4.2. *Let K be an ∞-category, and let $r\colon X \to K$ be a left fibration classified by a functor $G\colon K \to \mathcal{S}$. Then there is a canonical equivalence*

$$\Gamma(r) \xrightarrow{\simeq} \lim_K G,$$

functorial in G.

Proof. Under straightening, the left fibration r corresponds to G. Since straightening is an equivalence of ∞-categories, it identifies the mapping space

$$\Gamma(r) = \mathrm{Map}_{\mathrm{Cat}_{\infty,/K}}(\mathrm{id}_K, r)$$

with the mapping space in $\mathrm{Fun}(K, \mathcal{S})$ from the terminal object to G. By the universal property of the limit, the latter mapping space is canonically identified with $\lim_K G$. Functoriality is inherited from functoriality of straightening and the universal property of the limit functor. □

5 Finality in the slice

Fix an object $C \in \mathcal{C}$. Let

$$i_C \colon \{\mathrm{id}_C\} \hookrightarrow \mathcal{C}^{\mathrm{op}}_{/C}$$

denote the inclusion of the full simplicial subset spanned by the terminal object.

Proposition 5.1. *The functor i_C is final.*

Proof. By HTT, §4.1.3.1, it suffices to show that for each object $x \in \mathcal{C}^{\mathrm{op}}_{/C}$, the simplicial set

$$\{\mathrm{id}_C\} \times_{\mathcal{C}^{\mathrm{op}}_{/C}} (\mathcal{C}^{\mathrm{op}}_{/C})_{x/}$$

is weakly contractible. By the defining universal property of the overcategory, this pullback is canonically equivalent to the mapping space

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}_{/C}}(x, \mathrm{id}_C).$$

The latter is contractible by Proposition 3.1. Hence i_C is final. □

Corollary 5.2. *For every functor $G \colon \mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{S}$, restriction along i_C induces a canonical equivalence*

$$\lim_{\mathcal{C}^{\mathrm{op}}_{/C}} G \xrightarrow{\simeq} G(\mathrm{id}_C).$$

Proof. Since i_C is final, restriction along i_C induces an equivalence

$$\lim_{\mathcal{C}^{\mathrm{op}}_{/C}} G \xrightarrow{\simeq} \lim_{\{\mathrm{id}_C\}} G|_{\{\mathrm{id}_C\}}.$$

Since $\{\mathrm{id}_C\} \simeq \Delta^0$, the limit on the right is canonically equivalent to evaluation at id_C. □

6 Yoneda

The Yoneda lemma is now immediate from the preceding infrastructure.

Theorem 6.1 (Yoneda lemma)**.** *The bifunctor*

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}) \to \mathcal{S}, \qquad (C, F) \longmapsto \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), F),$$

is canonically equivalent to the evaluation bifunctor

$$\mathrm{ev} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}) \to \mathcal{S}, \qquad (C, F) \longmapsto F(C).$$

Equivalently, for each $C \in \mathcal{C}$ and each $F \in \mathcal{P}(\mathcal{C})$ there is a canonical natural equivalence

$$\Phi_{C,F} \colon \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), F) \xrightarrow{\simeq} F(C).$$

Proof. Fix $(C, F) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{P}(\mathcal{C})$. Let

$$p\colon \mathcal{E} \to \mathcal{C}^{\mathrm{op}}$$

be a left fibration classified by F, and let

$$q_C\colon \mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{C}^{\mathrm{op}}$$

be the representable left fibration classified by $j(C)$. Since $\mathrm{St}_{\mathcal{C}^{\mathrm{op}}}$ is an equivalence, it induces an equivalence on mapping spaces:

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), F) \simeq \mathrm{Map}_{\mathrm{LFib}(\mathcal{C}^{\mathrm{op}})}(q_C, p).$$

Because $\mathrm{LFib}(\mathcal{C}^{\mathrm{op}})$ is a full subcategory of $\mathrm{Cat}_{\infty,/\mathcal{C}^{\mathrm{op}}}$, the right-hand side may be computed in the ambient slice ∞-category.

Form the pullback square

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C}^{\mathrm{op}}_{/C} & \xrightarrow[q_C]{} & \mathcal{C}^{\mathrm{op}}. \end{array}$$

By Proposition 4.1, there is a canonical equivalence

$$\mathrm{Map}_{\mathrm{LFib}(\mathcal{C}^{\mathrm{op}})}(q_C, p) \simeq \Gamma(p').$$

Let

$$F'\colon \mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{S}$$

be the functor classified by p'. By Proposition 4.2,

$$\Gamma(p') \simeq \lim_{\mathcal{C}^{\mathrm{op}}_{/C}} F'.$$

By Corollary 5.2, restriction along i_C yields a canonical equivalence

$$\lim_{\mathcal{C}^{\mathrm{op}}_{/C}} F' \simeq F'(\mathrm{id}_C).$$

Therefore

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), F) \simeq F'(\mathrm{id}_C).$$

It remains to identify $F'(\mathrm{id}_C)$ with $F(C)$. Since F' is the straightening of p', the universal property of straightening identifies $F'(\mathrm{id}_C)$ with the fiber $\mathcal{E}'_{\mathrm{id}_C}$. On the other hand, by the universal property of the pullback square defining $\mathcal{E}'$, the fiber of p' at id_C is canonically equivalent to the fiber of p at the image of id_C under q_C, namely the object C of $\mathcal{C}^{\mathrm{op}}$. Thus there are canonical equivalences

$$F'(\mathrm{id}_C) \simeq \mathcal{E}'_{\mathrm{id}_C} \simeq \mathcal{E}_C.$$

Since p is classified by F, straightening identifies $\mathcal{E}_C$ with $F(C)$. Combining the preceding equivalences gives

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), F) \simeq F(C).$$

Each step is functorial in both variables: $C \mapsto q_C$ is functorial by the functoriality of the representable left fibration, $F \mapsto p$ is functorial by unstraightening, pullback is functorial in the diagram, the equivalence between sections and limits is functorial in the classified diagram, and restriction along the final functor i_C is functorial in the diagram. Hence the composite construction refines to an equivalence of bifunctors. □

Corollary 6.2. *The Yoneda embedding*

$$j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

is fully faithful. Equivalently, for all objects $C, D \in \mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), j(D))$$

is an equivalence of spaces.

Proof. Apply Theorem 6.1 with $F = j(D)$. One obtains

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(j(C), j(D)) \simeq j(D)(C) \simeq \mathrm{Map}_{\mathcal{C}}(C, D).$$

This equivalence is natural in (C, D), hence identifies the mapping-space bifunctor of $\mathcal{C}$ with that of the essential image of j. □

Remark 6.3. The theorem is not a calculation internal to the presheaf ∞-category. It is the assertion that the representable object $j(C)$ is geometrically realized by the slice left fibration $\mathcal{C}^{\mathrm{op}}_{/C} \to \mathcal{C}^{\mathrm{op}}$, and that the universal property of the terminal object id_C computes sections of its pullback against an arbitrary left fibration. The Yoneda equivalence is therefore the composite of straightening, pullback stability, finality, and the universal property of the limit.

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References

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